SPONTANEOUS FERMION PRODUCTION BY A SUPERCRITICAL POTENTIAL WELL*

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Abstract

A second quantised theory of electrons and positrons in a deep time-dependent potential well is discussed. It is shown that positron production from the well is a natural consequence of Dirac's hole theory when the strength of the well becomes supercritical. A formalism is developed whereby the amplitude for emission of a positron of a given momentum can be calculated. The difference between positron production and electron-positron pair production is demonstrated. Considerations of the vacuum charge and of Levinson's theorem are required for a full description of the problem.

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1. Introduction

Gershtein and Zeldovich [1] showed in 1969 that spontaneous positron production was possible when two heavy bare nuclei with total charge Z greater than some critical value Z_c collided with each other. Z_c is the value of Z for which the lowest 1S state of the hydrogenlike atom with total charge Z distributed on a nucleus of radius R has energy $E = -m$, where m is the electron mass. Pomeranchuk and Smorodinskii [2] calculated $Z_c = 200$ for a nucleus assuming the electrostatic potential is constant for r<R, while Greiner and Pieper [3] calculated $Z_c = 169$ for a homogeneous spherical charge distribution.

Zeldovich and Popov [4] subsequently reviewed the theoretical basis of this process. (For a recent overview of the problem and of other related topics the reader may find the monograph [5] useful.) They were able to set up a self-consistent picture of spontaneous positron production but from the viewpoint of a first-quantised Dirac theory. They state, however, that "at $Z > Z_c$ we have in principle a many-body problem. This constitutes the third stage of the investigation of the $Z > Z_c$ situation, and has not been completed as yet. In an exact formulation, it is necessary to consider the equations of the electron-positron wave field with creation and annihilation operators, and to use the second quantisation formalism". They go on to say that they predict the results of experiments on the basis of single particle solutions of the Dirac equation but that "such conclusions and predictions require a justification, which can be obtained only from the exact many-body theory"

This is the problem we have set ourselves in this paper for as far as we know it has not yet been tackled. For simplicity we have considered spontaneous fermionic production in the simplest supercritical external field we could think of, which is a sufficiently deep one-dimensional square well. We shall find that many of Zeldovich and Popov's results can indeed be justified. We shall also see that the preparation of the supercritical potential is an essential feature in the problem and that while it is true that a static supercritical square well does not radiate [6], it is also uninteresting because in any experimental situation it is necessary to create the supercritical potential from sub-critical potentials (by, for example, colliding two heavy ions together) and thus the time-dependence of the external field is an essential ingredient.

We first have to elucidate various questions relating to the Dirac equation in one dimension. In section 2 we consider a Dirac particle in a square well, write down the spectrum of bound states and give expressions for the scattering amplitude and the phase shift. We also present the conditions for criticality. In section 3 we give the second quantized version of the theory paying particular attention to possible conventions related to the labelling of states. We review the concept of the vacuum charge that plays a central role in this discussion. In section 4 we derive Levinson's theorem for the Dirac equation in one dimension and express the vacuum charge in

terms of phase shifts. In section 5 we illustrate some of the above concepts in the case of a δ function potential. In section 6 we apply the methods and results of the previous sections to the particular case of an external field which makes the transition from slightly subcritical to slightly supercritical to slightly subcritical and show in detail that fermionic radiation can occur. For this it is necessary to arrange for the ground state of the well to be initially empty, but we demonstrate how we end up with a filled bound state and a free fermion (which we identify for convenience with a positron). Second quantization and use of the Bogoliubov transformation are essential in this treatment.

2. The One Particle Dirac Equation in One Dimension

2.1. The Free Dirac Equation.

We adopt the convention $\gamma_0 = \sigma_3$, $\gamma_1 = i\sigma_1$. The above choice agrees with $\gamma_i\gamma_j + \gamma_j\gamma_i = 2g_{ij}$. The free Dirac Hamiltonian in one dimension is then

$$H_0 = -\sigma_2 p + \sigma_3 m$$

and the Dirac equation takes the form

$$(\sigma_1 \frac{\partial}{\partial x} - E\sigma_3 + m)\psi = 0 \tag{2.1}$$

In what follows $\mathbf{k}$ stands for the wavevector, k for its magnitude and $\varepsilon = |E| = +\sqrt{k^2 + m^2}$. We try a plane wave of the form

$$\begin{pmatrix} A \\ B \end{pmatrix} e^{ikx - iEt} \tag{2.2}$$

and substitute in (2.1). The equation is satisfied by $A = ik, B = E - m$ where $E = \pm\varepsilon$. Note that in one dimension the negative energy solution is obtainable from the positive energy one simply by replacing ε by $-\varepsilon$ unlike what happens in three dimensions. We normalize the 2 dimensional spinor in (2.2) by requiring $\int dx u^\dagger u = 1$. The positive energy solutions have the form

$$N_+(\varepsilon) \begin{pmatrix} ik \\ \varepsilon - m \end{pmatrix} e^{ikx - i\varepsilon t} \tag{2.3}$$

and the negative energy ones

$$N_-(\varepsilon) \begin{pmatrix} ik \\ -\varepsilon - m \end{pmatrix} e^{ikx + i\varepsilon t} \tag{2.4}$$

where $N_\pm(\varepsilon)$ are normalization factors. In what follows we will either consider the particle in a box of length $2L$ and imply periodic boundary conditions at $x = -L$ and $x = L$ or use continuum normalization. In the former case

$$N_+(\varepsilon) = \frac{1}{\sqrt{2L}\sqrt{2\varepsilon(\varepsilon - m)}} \ , \ N_-(\varepsilon) = \frac{1}{\sqrt{2L}\sqrt{2\varepsilon(\varepsilon + m)}}$$

and in the latter

$$N_+(\varepsilon) = \frac{1}{\sqrt{2\pi}\sqrt{2\varepsilon(\varepsilon - m)}} \ , \ N_-(\varepsilon) = \frac{1}{\sqrt{2\pi}\sqrt{2\varepsilon(\varepsilon + m)}}$$

Finally we quote the transformation of a wavefunction under parity:

$$\psi'(x,t) = \sigma_3 \psi(-x,t) \tag{2.5}$$

2.2. The Dirac Particle in a Square Well: Bound States.

Consider a square well of depth $-V$ extending from $x = -a$ to $x = a$. The electron Hamiltonian is

$$H_0 = -\sigma_2 p - V + \sigma_3 m$$

where V is the zeroth component of A_μ. The electron charge is taken to be -1. The Dirac equation reads

$$(\sigma_1 \frac{\partial}{\partial x} - (E + V)\sigma_3 + m)\psi = 0 \tag{2.6}$$

Define

$$p \equiv \sqrt{(E+V)^2 - m^2} \tag{2.7}$$

Inside the well try the wavefunction

$$\begin{pmatrix} A\cos px \\ B \sin px \end{pmatrix}$$

According to (2.5) this describes a state even under parity. Substituting in (2.1) we determine A, B (modulo a factor):

$$\psi_{even} = \begin{pmatrix} p\cos px \\ (E + V - m)\sin px \end{pmatrix} \tag{2.8}$$

for $-a \leq x \leq a$. For $x \geq a$ it can easily be checked that

$$\psi_{even} = s \begin{pmatrix} -\kappa \\ E - m \end{pmatrix} e^{-\kappa x} \tag{2.9}$$

with

$$\kappa = \sqrt{E^2 - m^2} \tag{2.10}$$

satisfies the Dirac equation. Matching (2.8) and (2.9) at $x = a$ and eliminating s we obtain the equation that determines the spectrum of the even bound states:

$$\tan ap = \sqrt{\frac{(m-E)(E+V+m)}{(m+E)(E+V-m)}} \tag{2.11}$$

Note that it is implicit in (2.7) that for a subcritical potential (i.e. $V < 2m$; see below) permissible bound states (even or odd) satisfy $E > m - V$. In a similar way we can determine the wavefunctions of the odd bound states. It turns out that for $-a \leq x \leq a$

$$\psi_{odd} = \begin{pmatrix} -p \sin px \\ (E+V-m)\cos px \end{pmatrix} \tag{2.12}$$

and for $x > a$

$$\psi_{odd} = s' \begin{pmatrix} -\kappa \\ E-m \end{pmatrix} e^{-\kappa x} \tag{2.13}$$

Matching (2.12) and (2.13) at $x = a$ and eliminating s' we obtain the equation that determines the spectrum of the odd bound states:

$$\tan ap = -\sqrt{\frac{(m+E)(E+V-m)}{(m-E)(E+V+m)}} \tag{2.14}$$

These results are well known and can be found, for example, in reference [3].

2.3. The Dirac Particle in a Square Well: Scattering States.

Consider a wave incident from the left. The corresponding wavefunction is

$$\begin{pmatrix} ik \\ E-m \end{pmatrix} e^{ikx} + B \begin{pmatrix} -ik \\ E-m \end{pmatrix} e^{-ikx} \tag{2.15}$$

for $x < a$,

$$C \begin{pmatrix} ip \\ E+V-m \end{pmatrix} e^{ipx} + D \begin{pmatrix} -ip \\ E+V-m \end{pmatrix} e^{-ipx} \tag{2.16}$$

for $-a \leq x \leq a$ and

$$F \begin{pmatrix} ik \\ E-m \end{pmatrix} e^{ikx} \tag{2.17}$$

for $x \geq a$. To calculate the wavefunction we require continuity at $x = -a$ and at $x = a$. The coefficients B, C, D, F are calculated in [7]. We only give the results for the phase shift. Define

$$F(E) = |F(E)|\, e^{i\delta(E)}$$

and

$$\gamma = \frac{k}{p}\frac{E+V+m}{E+m} \tag{2.18}$$

Then

$$\delta(E) = \arctan\left(\frac{1+\gamma^2}{2\gamma}\tan 2pa\right) - 2ka \tag{2.19}$$

In what follows the phase shift will sometimes be written with two arguments, i.e. $\delta(E,V)$, the second referring to the particular value of the potential. Whenever we write $\delta(E,0)$ we understand the limit $\delta(E,V\to 0)$. The algebra leading to (2.19) is right regardless of the sign of the energy. Recalling that $\epsilon \equiv |E|$ the phase shifts for positive and negative energy will be written as $\delta_+(\epsilon,V)$, $\delta_-(\epsilon,V)$ respectively. The phase shift is defined modulo a multiple of π. We choose

$$\delta(E,0) = 0 \tag{2.20}$$

(if $V=0$ then $k\sim p$, $\gamma\to 1$ and (2.19) satisfies (2.19)). We also write

$$\delta(\pm\infty,V) = \pm 2Va \tag{2.21}$$

Indeed if $E\to+\infty$ (with V,m fixed) then $k\sim E, p\sim E$, (2.18) yields $\gamma\sim 1$ and from (2.19) $\delta\sim 2pa-2ka\to 2Va$. Similarly for $E\to-\infty$, $\gamma\sim 1$ and $\delta\sim 2pa-2ka\to -2Va$. Note that (2.21) satisfies $\delta(\infty)+\delta(-\infty)=0$ as shown in 3 dimensions by Ma and Ni [8]). We also have

$$\delta(\pm\infty) = \mp\int_{-\infty}^{\infty} V(x)dx$$

which is analagous to the 3-dimensional result [8]

$$\delta(\pm\infty) = \mp\int_{0}^{\infty} V(r)dr$$

It must be emphasized that this result is in contrast to what happens in nonrelativistic quantum mechanics where the phase shift at high energy always tends to zero (modulo a multiple of π). The absence of multiples of π in the right hand side of (2.20), (2.21) is not a matter of convention. That would be the case had we specified e.g. $\delta(+\infty,V)$ only. Having however fixed the phase shift at one end of the energy range the rest should follow. It is better to regard (2.20), (2.21) as dictated by physical considerations as well. For $V\to 0$ we expect the results of the Dirac equation to coincide with those of the Schroedinger equation and (2.20), (2.21) satisfy this requirement (without any multiples of π). Notice that the negative energy phase shift makes sense in the context of nonrelativistic quantum mechanics as well: it corresponds to the scattering of a positron of momentum $-\mathbf{k}$.

We now wish to make a statement concerning the threshold values $\delta(\pm m,V)$. For a certain fixed V consider positive (negative) energy states very near threshold, i.e. $E\to m+$ ($E\to -m-$). Then the quantity γ defined in (2.18) has small absolute value and is positive (negative). Hence from (2.19) we conclude that

$$\delta(m,V) = \frac{\pi}{2} + n(V)\pi \tag{2.22}$$

$$\delta(-m,V) = -\frac{\pi}{2} + n'(V)\pi \tag{2.23}$$

where $n(V)$, $n'(V)$ are integers. Notice that at this stage no statement is made about $n(V)$, $n'(V)$. In particular the values at threshold $n(0)$, $n'(0)$ are unknown; see section 4 for details. Although the above equations have the look of Levinson's theorem in nonrelativistic quantum mechanics (see e.g. [9], p.354) in the case of the Dirac equation it is the sum $n + n'$ that is connected to the number of bound states [8]. We consider this point in more detail in section 4. However the occurrence of $\frac{\pi}{2}$ in (2.22) and (2.23) has the same origin as the $\frac{\pi}{2}$ in the corresponding equation in [9], namely the presence of a half-bound state in one-dimension [9].

We are particularly interested in the case where there is no reflection and so the transmission coefficient is a maximum. These we call, following Bohm [10], transmission resonances. Setting $B = 0$ and matching the above expressions at the boundaries we get four equations and eliminating C, D, F we obtain the equation determining the spectrum of the resonances

$$2pa = N\pi \tag{2.24}$$

This relation connecting wavelength and dimension of the well is familiar from the Fabry-Perot etalon. The similarity in the mathematical treatment of Fabry-Perot and a Dirac square well was first pointed out in reference [6]. We should emphasize, however, that transmission resonances are not resonances in the normal sense; the phase shift does not increase through $\pi/2$ as it does with a proper resonance.

A further point concerning the relation between bound states and transmission resonances may be in order. Consider a transmission resonance of energy E with k the incident wavevector and p the wavevector between the walls. Using the boundary conditions one can easily show that

$$\frac{C}{D} = e^{i2p'a} \tag{2.25}$$

The above equation together with (2.24) yield $C = \pm D$. Substituting in (2.16) we get that the transmission resonance wavefunction between the walls for $N = 1$ has the form (2.8). In other words if we consider an almost critical potential then the wavefunction of the lowest lying *(even)* bound state and the wavefunction between the walls of the upper ($N = 1$) resonance approach each other as the potential varies and the energies E_{bound} and E_{res} approach $-m$ from above and below respectively. Hence the overlap integral betwen the two wavefunctions is sizeable.

We define positive energy spinor wavefunctions $u_{(+)}(\mathbf{k}, x)$ given by (2.15), (2.16), (2.17) multiplied by $N_+(\varepsilon)$ and negative energy ones $u_{(-)}(\mathbf{k}, x)$ replacing E by $-E$ multiplied by $N_-(\varepsilon)$. Note that $u^\dagger_{(+)}u_{(-)} = u^\dagger_{(-)}u_{(+)} = 0$ and

$$\int dx u^\dagger_{(+)}(\mathbf{k}, x)u_{(+)}(\mathbf{k}', x) = \int dx u^\dagger_{(-)}(\mathbf{k}, x)u_{(-)}(\mathbf{k}', x) = \delta(\mathbf{k} - \mathbf{k}') \tag{2.26}$$

$u^\dagger(\mathbf{k},x) = u(-\mathbf{k},x)$ for both kinds of spinors. These wavefunctions refer to scattering states. We also have to include bound states. The spinor wavefunction corresponding to the jth bound state is denoted by $u_j(x)$ and can be read off (2.8), (2.9), (2.12), (2.13) (at present the notation does not distinguish between even and odd bound states). u_j is assumed to be normalized $\int dx u_j^\dagger u_j = 1$ and is orthogonal to the continuum wavefunctions.

Since parity is a good quantum number it is convenient to introduce spinor wavefunctions that transform under parity in a definite way. Using (2.5) define parity even and odd spinors corresponding to positive energies:

$$u_{(+)e}(k,x) \equiv \frac{1}{\sqrt{2}}\Big(u_{(+)}(\mathbf{k},x) - \sigma_3 u_{(+)}(-\mathbf{k},x)\Big) \tag{2.27}$$

$$u_{(+)o}(k,x) \equiv \frac{1}{\sqrt{2}}\Big(u_{(+)}(\mathbf{k},x) + \sigma_3 u_{(+)}(-\mathbf{k},x)\Big) \tag{2.28}$$

Replacing the subscript (+) by (-) we obtain the corresponding expressions for negative energy spinors.

2.4. Criticality

Let us now address the notion of criticality that plays a crucial role in the discussion. It is quite clear that for a small value of the potential equation (2.11) always has a solution corresponding to an even bound state. Suppose that the potential deepens gradually. At some stage the energy of the bound state crosses zero; the implications of this effect on the vacuum charge will be discussed later. For greater values of the potential the energy approaches $-m$ and for a critical value V_1^c (the subscript 1 refers to the fact that this is the first bound state that disappears) the bound state merges with the negative energy continuum states. Whether other bound states have appeared in the meantime is considered presently. The wavevector p between the walls is related to the potential by (2.7) and solving for V we get

$$V = \sqrt{p^2 + m^2} - E \tag{2.29}$$

It is clear from (2.11) that when

$$ap = \frac{\pi}{2} \tag{2.30}$$

the even bound state is at $-m$ and subsequently disappears. This corresponds to a critical value of the potential

$$V_1^c = \sqrt{\frac{\pi^2}{4a^2} + m^2} + m \tag{2.31}$$

Comparing (2.30) with (2.24) we see that the bound state wavefunction goes over to the $N = 1$ transmission resonance wavefunction. On the other hand it is clear from

(2.14) that when $ap = \dfrac{\pi}{2}$ a new *odd* bound state appears at $E = m$. This corresponds to the value of the potential

$$V_{odd1} = \sqrt{\frac{\pi^2}{4a^2} + m^2} - m \tag{2.32}$$

When $pa = \pi$ the second *even* bound state appears at

$$V_{even,2} = \sqrt{\frac{\pi^2}{a^2} + m^2} - m \tag{2.33}$$

Again for $pa = \pi$ the first *odd* bound state disappears corresponding to the potential

$$V_2^c = \sqrt{\frac{\pi^2}{a^2} + m^2} + m \tag{2.34}$$

and to the $N = 2$ transmission resonance. It should by now be clear how the pattern repeats itself.

3. Second Quantization.

Let N_+, N_- be the number of positive and negative bound states respectively. We expand ψ in terms of the continuous spectrum wavefunctions (2.27), (2.28) and of the bound states:

$$\begin{aligned}\psi(x,t) = \textstyle\sum_k \{a_e(k,0)u_{(+)e}(k,x)e^{-iEt} + a_o(k,0)u_{(+)o}(k,x)e^{-iEt} + \\ + c_e^\dagger(k,0)u_{(-)e}(k,x)e^{iEt} + c_o^\dagger(k,0)u_{(-)o}(k,x)e^{iEt}\} + \\ + \{\textstyle\sum_{j=1}^{N_+} b_j(0)u_j(x)e^{-iE_jt} + \sum_{j=1}^{N_-} d_j^\dagger(0)u_j(x)e^{-iE_jt}\}\end{aligned} \tag{3.1}$$

Operators $a^\dagger, a$ create and annihilate travelling electrons; $c^\dagger, c$ are the corresponding ones for positrons. Operators b_j ($b_j^\dagger$) annihilate (create) bound electrons whereas d_j ($d_j^\dagger$) annihilate (create) bound positrons. The use of the ($\dagger$) in (3.1) is dictated by the sign of the exponential and conforms to current literature. The Hermitean conjugate expansion is

$$\begin{aligned}\psi^\dagger(x,t) = \textstyle\sum_k \{a_e^\dagger(k,0)u_{(+)e}^\dagger(k,x)e^{iEt} + a_o^\dagger(k,0)u_{(+)o}^\dagger(k,x)e^{iEt} + \\ + c_e(k,0)u_{(-)e}^\dagger(k,x)e^{-iEt} + c_o(k,0)u_{(-)o}^\dagger(k,x)e^{-iEt}\} + \\ + \{\textstyle\sum_{j=1}^{N_+} b_j^\dagger(0)u_j^\dagger(x)e^{iE_jt} + \sum_{j=1}^{N_-} d_j(0)u_j(x)e^{iE_jt}\}\end{aligned} \tag{3.2}$$

where we took into account the reality of u_j. The standard anticommutation relations are obeyed

$$\begin{aligned}\{a_{e,o}(k,t), a_{e,o}^\dagger(k',t\} = \delta_{kk'}\delta_{eo}\ ,\ \{b_i(t), b_j^\dagger(t)\} = \delta_{ij} \\ \{c_{e,o}(k,t), c_{e,o}^\dagger(k',t\} = \delta_{kk'}\delta_{eo}\ ,\ \{d_i(t), d_j^\dagger(t)\} = \delta_{ij}\end{aligned} \tag{3.3}$$

We work in the Heisenberg picture throughout: The time dependence is carried by operators whereas state vectors are time independent. However basis ket vectors (and in particular the vacuum) are time dependent (see e.g. [11]). The vacuum $|0\rangle$ is defined by

$$a_{e,o}(k)\,|0\rangle = c_{e,o}(k)\,|0\rangle = b_i\,|0\rangle = d_i\,|0\rangle = 0 \tag{3.4}$$

It must be stressed however that the definition of the vacuum is to some extent a matter of convention. A complete statement on the physics of the problem is made when one defines both the vacuum and also the state vector of the system. We will elaborate this point later on.

The total charge is defined by (N.B. according to our conventions the electron charge is -1)

$$Q(t) \equiv \int dx \rho(x,t) = -\frac{1}{2}\int dx \left[\psi^\dagger(x,t), \psi(x,t)\right] \tag{3.5}$$

Substituting (3.1), (3.2) and using (3.3) we get

$$Q = Q_{norm} + Q_0 \tag{3.6}$$

where

$$\begin{gathered} Q_{norm} \equiv \textstyle\sum_k \{-a_e^\dagger(k,t)a_e(k,t) + c_e^\dagger(k,t)c_e(k,t) - a_o^\dagger(k,t)a_o(k,t) + \\ + c_o^\dagger(k,t)c_o(k,t) - \textstyle\sum_{j=1}^{N_+} b_j^\dagger b_j + \sum_{j=1}^{N_-} d_j^\dagger d_j\} \end{gathered} \tag{3.7}$$

and

$$Q_0 \equiv \frac{1}{2}\left\{\sum_k (\text{states with } E>0) - \sum_k (\text{states with } E<0)\right\} \tag{3.8}$$

Expressions (3.7), (3.8) are rather formal. To be precise we have to calculate phase shifts and density of states and transform the sums to integrals. This is done in section 4 and illustrated in section 5. Given the definition (3.4) of the vacuum we immediately get

$$\langle 0| Q |0\rangle = Q_0 \tag{3.9}$$

i.e. the vacuum charge turns out to be the spectral asymmetry of the Hamiltonian. Note that we would not have obtained the connection between Q_0 and the vacuum asymmetry had we not identified bound states with $E<0$ as positrons or not used the commutator in (3.5). Note also that Q_0 clearly depends on the potential. The same applies to Q_{norm} although the notation does not indicate this. The wavefunctions that appear in (3.1), (3.2) depend on the strength of the potential. By implication the same is true for the creation and annihilation operators and thus for Q_{norm}.

To see precisely how the above formalism should be used let us consider the electron field interacting with a time dependent potential $V(x,t)$. The Hamiltonian is given by

$$\mathcal{H} = \mathcal{H}(freeDirac) - \frac{1}{2}\int dx V(x,t)\left[\psi^\dagger(x,t), \psi(x,t)\right] \tag{3.10}$$

It is quite clear that (3.10) conserves Q. In order to avoid extraneous complications let us assume for the moment that the potential is relatively weak and that there is only one bound state at energy E_b slightly above zero , i.e. $N_+ = 1, N_- = 0$. Denote by $Q_{0,init}$ the initial vacuum charge. In the present case (3.1) reads

$$\begin{gathered} \psi(x,t) = \textstyle\sum_k \{a_e(k,0)u_{(+)e}(k,x)e^{-iEt} + a_o(k,0)u_{(+)o}(k,x)e^{-iEt} + \\ + c_e^\dagger(k,0)u_{(-)e}(k,x)e^{iEt} + c_o^\dagger(k,0)u_{(-)o}(k,x)e^{iEt}\} + b(0)u(x)e^{-iE_bt} \end{gathered} \tag{3.11}$$

Before going any further we have to specify the state $|S\rangle$ of the system. We take all electron and positron scattering states to be empty and the bound state is also unfilled (in other words this state coincides with the vacuum state as defined in (3.4). Then

$$a_{e,o}(k)\,|S\rangle = c_{e,o}(k)\,|S\rangle = 0 \tag{3.12}$$

$$b\,|S\rangle = 0 \tag{3.13}$$

According to (3.12) $\langle S|\,Q_{norm}\,|S\rangle = 0$ and so

$$\langle S|\,Q\,|S\rangle = Q_{0,init} \tag{3.14}$$

We now slowly increase the potential strength until eventually the bound state energy is found below$E = 0$. We denote everything pertaining to this later time by a ($\hat{}$). According to (3.1) the field now evolves as

$$\begin{aligned}\hat{\psi}(x,t) = \textstyle\sum_k \{\hat{a}_e(k,0)\hat{u}_{(+)e}(k,x)e^{-i\hat{E}t} + \hat{a}_o(k,0)\hat{u}_{(+)o}(k,x)e^{-i\hat{E}t} + \\ +\hat{c}_e^\dagger(k,0)\hat{u}_{(-)e}(k,x)e^{i\hat{E}t} + \hat{c}_o^\dagger(k,0)\hat{u}_{(-)o}(k,x)e^{i\hat{E}t}\} + \hat{d}^\dagger(0)\hat{u}(x)e^{i\hat{E}_b t}\end{aligned} \tag{3.15}$$

Since the change in potential strength is assumed to be sufficiently slow and small and since there was no electron occupying the bound state initiallly, it is clear that there will still be no electron occupying the bound state.Nevertheless, by the conventional interpretation of Dirac's theory, the absence of an electron in a state with $E < 0$ signifies the presence of a positron which in this case is a bound positron. For a slow enough change in the potential strength it is also clear that no radiation is emitted. Thus in terms of the ($\hat{}$) operators the state $|S\rangle$ satisfies

$$\hat{a}_{e,o}(k)\,|S\rangle = \hat{c}_{e,o}(k)\,|S\rangle = 0$$

$$\hat{d}^\dagger\hat{d}\,|S\rangle = |S\rangle \tag{3.16}$$

Despite the appearance of a bound positron total charge is conserved, i.e. $\langle S|\,Q\,|S\rangle$ is still equal to $Q_{0,init}$ when Q is expressed in terms of the ($\hat{}$) operators. To see this note that the $\hat{d}^\dagger\hat{d}$ term in Q_{norm} (3.7) now contributes a term $+1$ (the positron charge) because of (3.16). This however is cancelled by a term -1 originating from Q_0 simply by counting states; since one bound state has now dived below zero the vacuum asymmetry is equal to $Q_{0,init} - 1$.

Finally it may be of interest to examine the expectation value of the total charge when, as the potential changes, the energy of the bound state exactly reaches zero. We can then use either expansion

$$\begin{aligned}\psi(x,t) = \textstyle\sum_k \{a_e(k,0)u_{(+)e}(k,x)e^{-iEt} + a_o(k,0)u_{(+)o}(k,x)e^{-iEt} + \\ +c_e^\dagger(k,0)u_{(-)e}(k,x)e^{iEt} + c_o^\dagger(k,0)u_{(-)o}(k,x)e^{iEt}\} + bu(x)\end{aligned} \tag{3.17}$$

or

$$\psi(x,t) = \sum_k \{a_e(k,0)u_{(+)e}(k,x)e^{-iEt} + a_o(k,0)u_{(+)o}(k,x)e^{-iEt} + \\ + c_e^\dagger(k,0)u_{(-)e}(k,x)e^{iEt} + c_o^\dagger(k,0)u_{(-)o}(k,x)e^{iEt}\} + d^\dagger u(x) \tag{3.18}$$

The use of b or $d^\dagger$ in (3.17) and (3.18) respectively is purely a matter of taste since in the absence of an energy exponential our convention dictates nothing. Assuming again that the bound state level has an electron vacancy (3.17) entails

$$b\,|S\rangle = 0 \tag{3.19}$$

whereas (3.18) entails

$$d^\dagger d\,|S\rangle = |S\rangle \tag{3.20}$$

In both cases scattering states are vacant

$$a_{e,o}(k)\,|S\rangle = c_{e,o}(k)\,|S\rangle = 0$$

Notice however that the two conventions have different implications for the value of Q_0 when the level is at exactly zero energy. According to (3.19) the zero mode is counted as an electron state hence Q_0 of (3.8) gets a $+\frac{1}{2}$ contribution due to normal ordering of b and $b^\dagger$. In contrast according to (3.20) the zero mode is counted as a positron state hence Q_0 gets a $-\frac{1}{2}$ contribution due to normal ordering of d and $d^\dagger$. However physical results do not change. When we calculate $\langle S|\,Q_{norm}\,|S\rangle$ then according to the first convention there is no contribution from the zero mode because of (3.19). On the other hand according to the second convention there is a $+1$ contribution because of (3.20) and the presence of the $d^\dagger d$ term in Q_{norm}. Thus the total contribution to $\langle S|\,Q\,|S\rangle$ due to the zero mode is $+\frac{1}{2}$ as before. Notice that when the bound state is slightly above zero then its contribution to Q_0 is $+\frac{1}{2}$ (according to (3.8) and its contribution to $\langle S|\,Q_{norm}\,|S\rangle$ is zero according to (3.13). Thus the contribution of the bound state to $\langle S|\,Q\,|S\rangle$ depends smoothly on the potential.

4. Phase Shifts at Threshold and Levinson's Theorem and the Vacuum Charge.

We now discuss Levinson's theorem [13], [14] , [15], [16], for a one-dimensional system since it is intimately related to the discussion of the vacuum charge on which our interpretation of positron radiation is based. We follow the strategy used by Barton [12] in his discussion of Levinson's theorem for the Schroedinger equation in one dimension and enclose the system in a box of length $2L$ with periodic boundary conditions

$$\psi(-L) = \psi(L) \tag{4.1}$$

(Note that if one compares what follows with reference 12 one should interchange the roles of integers n and v.) We consider even and odd states separately. An odd state of positive energy has asymptotically the form

$$\psi_{odd}(x \to \pm\infty) = \begin{pmatrix} ik\sin(kx \pm \Delta_{o+}) \\ (\epsilon - m)\cos(kx \pm \Delta_{o+}) \end{pmatrix} \tag{4.2}$$

and an even one

$$\psi_{even}(x \to \pm\infty) = \begin{pmatrix} ik\cos(kx \pm \Delta_{e+}) \\ (\epsilon - m)\sin(kx \pm \Delta_{e+}) \end{pmatrix} \tag{4.3}$$

where we neglect normalization factors (the subscript $\pm$ denotes the energy sign). Negative energy states are obtained by replacing ϵ by $-\epsilon$ as already mentioned in section 1. The phase shifts introduced in section 2 are connected to the above by

$$\delta_+(\epsilon) = \Delta_{e+}(\epsilon) + \Delta_{o+}(\epsilon) \tag{4.4}$$

$$\delta_-(\epsilon) = \Delta_{e-}(\epsilon) + \Delta_{o-}(\epsilon) \tag{4.5}$$

Condition (4.1) implies

$$k(\nu_{e,\pm})L + \Delta_{e\pm}(k_\nu) = \nu_{e,\pm}\pi \tag{4.6}$$

$$k(\nu_{o,\pm})L + \Delta_{o\pm}(k_\nu) = \nu_{o,\pm}\pi \tag{4.7}$$

ν being integer. It is clear from (4.2), (4.3) that the wavevectors k are strictly positive; this will impose restrictions on ν. In the continuum limit calculations similar to the ones outlined in section 2 yield

$$\Delta_e(\pm m) = \frac{\pi}{2} + n_{e,\pm}(0)\,\pi \tag{4.8}$$

$$\Delta_o(\pm m) = n_{o,\pm}(0)\,\pi \tag{4.9}$$

(the argument in the above relations indictes that $V \to 0$). In the free case relations (4.6), (4.7) are still valid with no Δ appearing. For V small enough, the system reduces to that of the Schroedinger equation discussed in [12]. In that case Barton shows that for counting purposes it is sufficient to consider the number of positive energy states for a free particle to be integers $\nu_{e,+}$, $\nu_{o,+}$ ranging from a minimum value

$$\nu_{\min,free} = 1 \tag{4.10}$$

to some large cut-off value N. We repeat the process for negative energy states.We thus have $2N$ even solutions and $2N$ odd ones (counting both positive and negative energy states). Suppose that we now switch on the potential keeping it arbitrarily small. We know that the number of positive energy even scattering states decreases by 1 since there is one even bound state appearing. We have the same number of odd scattering states and the same number of negative energy scattering states. The total number of states is conserved. In other words

$$\nu_{\min,o,-} = \nu_{\min,o,+} = \nu_{\min,e,-} = 1 \tag{4.11}$$

$$\nu_{\min,e,+} = 2 \tag{4.12}$$

It should also be observed that for any value of $\nu_{\min}$ the wavevector $k(\nu_{\min})$ goes to zero in the large L limit. We are thus entitled to replace $\Delta_{e,o}(k(\nu_{\min,\pm}))$ by the appropriate threshold value $\Delta(\pm m)$. Substituting (4.8) in (4.6) we get

$$k(\nu_{e,+})\,L = -\frac{\pi}{2} - n_{e,+}(0)\pi + \nu_{e,+}\pi \tag{4.13}$$

$$k(\nu_{e,-})\,L = -\frac{\pi}{2} - n_{e,-}(0)\,\pi + \nu_{e,-}\pi \tag{4.14}$$

Similarly

$$k(\nu_{o,+})\,L = -n_{o,+}(0)\,\pi + v_{o,+}\pi \tag{4.15}$$

$$k(\nu_{o,-})\,L = -n_{o,-}(0)\,\pi + \nu_{o,-}\pi \tag{4.16}$$

From (4.11), (4.12) and from the fact that k is strictly positive we deduce that

$$n_{e,+}(0) = 1 \tag{4.17}$$

$$n_{e,-}(0) = 0 \tag{4.18}$$

In other words

$$\Delta_e(m) = \frac{3\pi}{2}, \Delta_e(-m) = \frac{\pi}{2}$$

Via a similar argument

$$n_{o,+}(0) = n_{o,-}(0) = 0$$

and

$$\Delta_o(m) = \Delta_o(-m) = 0 \tag{4.19}$$

This concludes the question of the determination of phase shifts at threshold mentioned after (4.5). Note that $\Delta_e(\pm m)$ has a discontinuity of $\frac{\pi}{2}$ at $V = 0$ just as it does for the one dimensional Schroedinger equation since an attractive one dimensional potential no matter how weak always has at least one bound state.

We can now turn on the potential gradually: then for specific values of V (cf. the discussion at the end of section 2) a scattering state (alternatively even or odd) crosses m and becomes bound. Since one scattering state is lost it is clear that $\nu_{\min,e,+}$ or $\nu_{\min,o,+}$ respectively increases by 1. From the requirement that k be positive we deduce from (4.11) and (4.17) that $n_{e,+}(V)$ and $\mathrm{n}_{o,+}(V)$ denote the number of even and odd states respectively that have crossed m for the given value of the potential (including the even bound state that exists just below m for an arbitrarily small value of the potential). Similarly as the potential deepens further, bound states cross $-m$ and join the continuum. The fact that more negative energy scattering states become available means that $\nu_{\min,e,-}$ and $\nu_{\min,o,-}$ decrease and from (4.18), (4.16) we deduce that $n_{e,-}$ and $n_{o,-}$ decrease. Since (cf. (4.16), (4.18)) they start at zero for vanishing potential we conclude that $n_{e,-}$ and $n_{o,-}$ are negative and that the absolute values $|n_{e,-}(V)|$ and $|n_{o,-}(V)|$ represent the number of even and odd (ex) bound states that have crossed $-m$ for the particular value of V. Simple bookkeeping then yields that $n_{e,+}(V) + n_{e,-}(V)$ and $n_{o,+}(V) + n_{o,-}(V)$ denote the number of even and odd bound states respectively. From (4.8) and (4.9) we get

$$n_{bound,even} = \frac{1}{\pi}\left(\Delta_e(m,V) + \Delta_e(-m,V)\right) - 1 \tag{4.20}$$

$$n_{bound,odd} = \frac{1}{\pi}\left(\Delta_o(m,V) + \Delta_o(-m,V)\right) \tag{4.21}$$

The above relations constitute Levinson's theorem in the present problem.

The connection between the number of bound states and the jumps of phase shifts at threshold by π can be seen directly in the case of the square well examined in section 2. Return to expression (2.19) for the phase shift, set $k \sim 0, E \sim m$ so that the quantity $\frac{1}{\gamma}$ in (2.19) tends to infinity regardless of the strength of the potential and vary V. For some particular value V_{new} a new bound state crosses $E = m$. Recall that according to the analysis of section 2.4 $\tan 2pa$ as a function of V vanishes at

V_{new}. From (2.7) it is also clear that tan $2pa$ is an increasing function of V, so since V is increasing, tan $2pa$ is crosses zero from *below*. Thus according to (2.19)

$$\delta\left(m,\, V_{new}-0\right)=\arctan\left(-\infty\right)$$

$$\delta\left(m,\, V_{new}+0\right)=\arctan\left(\infty\right)$$

hence when a new bound state appears $\delta\left(m\right)$ jumps by π, i.e. n in (2.22) increases by 1. In other words n counts the number of bound states that appear (including the bound state that exists just below m for an arbitrarily small V). By a similar reasoning when a bound state crosses zero and disappears (for *increasing* V), $\delta\left(-m\right)$ jumps by $-\pi$, i.e. n' decreases by 1. Hence for an attractive potential n' is negative and its absolute value counts the number of bound states that disappeared.

Before we move to the determination of the vacuum charge let us point out the constraints that (4.7), (4.9) impose on the minimum value of the wavevector k (for L large but finite). Setting $\Delta=0$ in the above relations we get that for a free field in all cases (positive or negative energy, even or odd parity)

$$k_{\min,free}=\frac{\pi}{L} \tag{4.22}$$

corresponding to $\nu=1$. For a very weak potential (4.7), (4.9) in conjunction with (4.15), (4.16), (4.19) yield (again in all cases)

$$k_{\min}=\frac{\pi}{2L} \tag{4.23}$$

To obtain the expression for the vacuum charge we convert sums over states using the standard expression that follows from (4.2), (4.3)

$$\sum(states)=\frac{1}{\pi}\int^{\infty}dk\left(L+\frac{d\delta}{dk}\right) \tag{4.24}$$

where the lower limit will be stated presently. Separating the contributions of odd and even states (3.8) gives

$$\begin{aligned} Q_0 \;=\; & \frac{1}{2}\{\frac{1}{\pi}\int_{k_{\min}}^{\infty}dk\left(L+\frac{d\Delta_{e+}(E)}{dk}\right)+n_{bound,even,+} \\ & +\frac{1}{\pi}\int_{k_{\min}}^{\infty}dk\left(L+\frac{d\delta_{o+}(-E)}{dk}\right)+n_{bound,odd,+}-\frac{1}{\pi}\int_{k_{\min}}^{\infty}dk\left(L+\frac{d\Delta_{e-}(E)}{dk}\right)- \\ & -n_{bound,even,-}-\frac{1}{\pi}\int_{k_{\min}}^{\infty}dk\left(L+\frac{d\Delta_{o-}(e)}{dk}\right)-n_{bound,odd,-} \end{aligned} \tag{4.25}$$

with $k_{\min}$ given by (4.23). It should be noticed that although $k_{\min}$ tends to zero in the $L\to\infty$ limit this is counterbalanced in the integral by the factor L. Although this is

irrelevant for our purposes it is in general essential if we want to ensure conservation of the total number of states. The fact that the lower limits of integration in (4.25) are all identical allows the L proportional terms to cancel out after replacing ∞ by a cut-off wavevector K (this is true even after a more careful regularization. The dk integration over the Δ functions is performed trivially and brings in the functions $\delta_+(\epsilon)$, $\delta_-(\epsilon)$ via (4.4), (4.5). For convenience we define the number of bound states according to the energy sign and regardless of parity

$$N_+ \equiv n_{bound,even,+} + n_{bound,odd,+}$$

$$N_- \equiv n_{bound,even,-} + n_{bound,odd,-}$$

Then Levinson's theorem can be trivially rewritten

$$N_+ + N_- = \frac{1}{\pi}\left(\delta_+(m) + \delta_-(m)\right) - 1$$

We finally get

$$Q_0 = \frac{1}{2}\left\{\frac{1}{\pi}\left(\delta_+(\infty, V) - \delta_+\left(m, V\right) - \delta_-(\infty, V) + \delta_-(m, V)\right) + N_+ - N_-\right\} \tag{4.26}$$

In the particular case of a square well (2.21), (4.26) yield

$$Q_0 = \frac{1}{2}\left\{\frac{4Va}{\pi} + \frac{1}{\pi}\left(-\delta_+\left(m, V\right) + \delta_-\left(m, V\right)\right) + N_+ - N_-\right\} \tag{4.27}$$

It may be appropriate at this point to enlarge on the role played by $k_{\min}$ introducted in (4.22), (4.23) and explain the meaning of the statement following (4.25). Suppose that the potential is extremely weak and that we have one (just) bound state. It is then natural to assume that the total number of positive energy states (scattering plus bound) is the same as before. Equation (4.24) gives

$$\begin{aligned}\sum\left(\textit{positive states}, V \to 0\right) \;&=\; \frac{L}{\pi}\int_{k_{\min}}^{K} dk + \frac{1}{\pi}\int_{k_{\min}}^{K} dk \frac{d\delta}{dk} + 1 = \\ &\frac{Lk}{\pi} - \frac{L}{\pi}k_{\min} + \frac{1}{\pi}\delta\left(k_{\min}\right) + 1\end{aligned} \tag{4.28}$$

For a sufficiently weak potential $\delta\left(K\right)$ can be neglected. As already explained for large $L\delta\left(k_{\min}\right) \sim \delta\left(0\right) = \dfrac{3\pi}{2}$. If we were to neglect the role of $k_{\min}$ (i.e. set $k_{\min} = 0$) we would get

$$\sum(\textit{positive states}, V \to 0) = \frac{LK}{\pi} - \frac{1}{2} \tag{4.29}$$

Given the first term in (4.29) would be there even in the free case it is clear that the above result and the idea of dropping $k_{\min}$ are absurd: (4.29) implies that switching on the potential creates *half* a state. However if we take (4.23) into account we get

$$\sum(\textit{positive states, } V \to 0) = \frac{LK}{\pi} - \frac{L}{\pi}.\,\frac{\pi}{2L} - \frac{1}{\pi}.\frac{3\pi}{2} + 1 = \frac{LK}{\pi} - 1 \tag{4.30}$$

On the other hand for *strictly zero* potential strength $k_{\min}$ is given by (4.22), hence

$$\sum(\textit{positive states, } V = 0) = \frac{LK}{\pi} - \frac{L}{\pi}.\,\frac{\pi}{L} - \frac{\pi}{L} = \frac{LK}{\pi} - 1$$

in manifest agreement with (4.30).

Let us focus on the behaviour of Q_0 for small changes of the potential. When the potential changes slightly and a new bound state appears the phase shifts $\delta(\pm\infty) = \pm 2Va$ change smoothly. The jump of $\delta(m)$ by π in (4.26) is counterbalanced by the increase in N_+ by 1. Thus Q_0 also changes smoothly. On the other hand if for a slight change of the potential a bound state crosses zero then the phase shifts behave smoothly, N_+ decreases by 1 and N_- increases by 1. Hence Q_0 decreases abruptly by 1. This however should not necessarily be construed as a physical discontinuity. Recall for example the argument after (3.16): For the particular state $|S\rangle$ considered there the decrease in Q_0 is counterbalanced by a corresponding increase in $\langle S|\, Q_{norm}\, |S\rangle$ since the empty bound electron state now appears as an occupied positron state with charge +1. Thus the total charge has no discontinuity.

Let us see this last point in somewhat greater detail. Suppose that one discrete level (the lowest lying even state in the present case) crosses zero for $V = V_0$. (For definiteness assume that a is such that the first odd state has already appeared; it can be checked that this is indeed feasible for say $a = .7$.) Then at $V = V_0-$ the even bound state is just above zero and we assume it to be empty. The vacuum charge is obtained from (4.27) with $N_+ = 2$, $N_- = 0$, $n = 1$, $n' = 0$

$$Q_0 = \frac{2V_0 a}{\pi} \tag{4.31}$$

and the expectation value of the total charge is

$$\langle S|\, Q(V_0-)\, |S\rangle = \langle S|\, Q_{norm}(V_0-)\, |S\rangle + Q_0$$

where the argument (V_0-) serves to remind that creation and annihilation operators depend on V. Consider a very small change in the potential so that $V = V_0+$. Since the even bound state has dived below zero it is now a positron state and according to the hole theory (and to our conventions) the absence of an electron amounts to the presence of a positron. So in terms of operators pertaining to $V = V_0+$ the state $|S\rangle$ defined by (3.12) satisfies

$$d^\dagger(V_0+)d(V_0+)\, |S\rangle = |S\rangle \tag{4.32}$$

Since the potential changed infinitesimally no radiation has been emitted. Hence according to (3.7)

$$\langle S|\, Q_{norm}(V_0+)\,|S\rangle = \langle S|\, Q_{norm}(V_0-)\,|S\rangle + 1$$

where the last term is due to the presence of $d^\dagger_{even}d_{even}$ in Q_{norm} and simply reflects the fact that the positron carries charge $+1$. The important point is that charge is conserved. Indeed now $N_+ = 1, N_- = 1$ and according to (**??**) Q_0 has decreased by 1:

$$Q_0 = \frac{2V_0 a}{\pi} - 1 \tag{4.33}$$

The above reflects the connection between vacuum charge and spectral asymmetry. The charge expectation value $\langle S|\, Q\,|S\rangle$ rests unchanged during the crossing.

5. An Example: The δ function Potential.

We consider the attractive (for electrons) potential $-\lambda\delta(x)$. We define this potential as a square well in the limit $V \to \infty, a \to 0$ so that Va is kept finite and $\lambda = 2Va$. (This definition avoids the problems of definition discussed in reference [14]. The advantage of the δ function limit lies in the fact that we can obtain simpler closed formulae for some of the quantities treated previously.

We first turn to the determination of the scattering amplitude and the phase shift. For a wave incident from the left

$$\psi(x<0) = \begin{pmatrix} ik \\ E-m \end{pmatrix} e^{ikx} + R \begin{pmatrix} -ik \\ E-m \end{pmatrix} e^{-ikx} \tag{5.1}$$

$$\psi(x>0) = F \begin{pmatrix} ik \\ E-m \end{pmatrix} e^{ikx} \tag{5.2}$$

A solution to (2.6) is clearly furnished by

$$\psi(x) = \int_{x_0}^{x} dx \exp(i\sigma_2(E + \lambda\delta(x) - m))\psi(x_0) \tag{5.3}$$

for some arbitrary x_0. Applying the above equation for $x_0 = 0-, x = 0+$ we connect (5.1), (5.2)

$$\psi(0+) = e^{i\lambda\sigma_2}\psi(0-) \tag{5.4}$$

or

$$\begin{pmatrix} ik \\ E-m \end{pmatrix} + R \begin{pmatrix} -ik \\ E-m \end{pmatrix} = (\cos\lambda + i\sin\lambda\sigma_2) F \begin{pmatrix} ik \\ E-m \end{pmatrix}$$

We thus get two equations for F, R. After trivial algebra

$$F(E) = \frac{k}{k\cos\lambda - iE\sin\lambda}$$

This leads to

$$\delta(E) = \arctan\left(\frac{E}{k}\tan\lambda\right) \tag{5.5}$$

Let us compare the above result to (2.19). To this end the $V \to \infty$ limit in section 2 should be taken from the start. Then clearly $2pa = \lambda$. Again in the same limit and after some algebra we get

$$\frac{1+\gamma^2}{2\gamma} \to \frac{E}{k}$$

Thus (2.19) and (5.5) agree. In the high energy limit clearly

$$\delta(\pm\infty) = \pm\lambda \tag{5.6}$$

(note that this agrees with the convention (2.20)). It also agrees with result (2.21). At threshold we get

$$\delta(m) = \frac{\pi}{2} + n(\lambda)\pi \tag{5.7}$$

$$\delta(-m) = -\frac{\pi}{2} + n'(\lambda)\pi \tag{5.8}$$

The analysis of the previous section still holds and entails $n(0) = n'(0) = 0$ at threshold.

We now turn to the spectrum of the bound states. The wavefunctions are of the form

$$\begin{pmatrix} -\kappa \\ E-m \end{pmatrix} e^{-\kappa|x|}$$

modulo a sign depending on parity. Applying (5.4) we get for even bound states

$$\begin{pmatrix} -\kappa \\ E-m \end{pmatrix} = \begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} -\kappa \\ E-m \end{pmatrix} \tag{5.9}$$

and for odd bound states

$$\begin{pmatrix} -\kappa \\ E-m \end{pmatrix} = -\begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} -\kappa \\ E-m \end{pmatrix} \tag{5.10}$$

Using relation (2.10) between κ and E we get the equations determining the spectrum. For even bound states

$$\tan\frac{\lambda}{2} = \sqrt{\frac{m-E}{m+E}} \tag{5.11}$$

and for odd bound states

$$\tan\frac{\lambda}{2} = -\sqrt{\frac{m+E}{m-E}} \tag{5.12}$$

Equations (5.11), (5.12) are identical to (2.11), (**??**) in the $V \to \infty$ limit. Both give the equation for the spectrum

$$E = m\cos\lambda \,\mathrm{sign}(\sin\lambda) \tag{5.13}$$

Thus even bound states appear when $\frac{\lambda}{2} = N\pi$ and disappear when $\frac{\lambda}{2} = \frac{\pi}{2} + N\pi$. Odd bound states appear when $\frac{\lambda}{2} = -\frac{\pi}{2} + N\pi$ and disappear when $\frac{\lambda}{2} = N\pi$. A distinctive feature of the δ function potential is that whenever a bound state of a certain parity appears a state of the opposite parity disappears (and vice versa). For example when $\lambda = \pi$ the original (even) bound disappears and the first odd one appears, at $\lambda = 2\pi$ the odd state disappears and the second even state appears etc.. In other words $\lambda \mathrm{mod}\pi$ counts the number of bound states (excluding the first one that

exists for arbitrarily small λ) that have crossed $E = m$. These results may also be obtained from the analysis of section 2.4 in the limit $a \to 0$ (thus essentially dropping m everywhere). Finally from (5.13) we deduce that a discrete level crosses zero when $\lambda = \frac{\pi}{2} + N\pi$.

Let us examine the vacuum charge in the δ function case. Suppose that N bound states (on top of the original half bound state existing for arbitrarily small λ) have crossed m, i.e. $\lambda = N\pi + \lambda'$, $0 \leq \lambda' < \pi$. Then $\delta(\infty) = \lambda$, $\delta(-\infty) = -\lambda$, $\delta(m) = \frac{3\pi}{2} + N\pi$, $\delta(-m) = -\frac{\pi}{2} - N\pi$. If the bound state is above zero, i.e. $N_+ = 1$, $N_- = 0$ (4.26) gives

$$Q_0 = \frac{\lambda'}{\pi} \tag{5.14}$$

In the same way if $N_+ = 0$, $N_- = 1$

$$Q_0 = \frac{\lambda'}{\pi} - 1 \tag{5.15}$$

Thus at $\lambda = \frac{\pi}{2}$ (when the crossing takes place) the vacuum carge jumps from $+\frac{1}{2}$ to $-\frac{1}{2}$, the difference being correctly equal to 1. As explained towards the end of section 3.1 the value of the vacuum charge *at* $\lambda = \frac{\pi}{2}$ depends on our convention concerning the labelling of the zero mode.

6. The Transition from a Subcritical to a Supercritical Potential and Positron Emission.

We now finally consider the transition from a subcritical to a supercritical potential. We assume that the potential starts at a slightly subcritical value V_{sub}, makes an abrupt change to a supercritical value $V_{\sup er}$ at time t_1, stays fixed at this value until time t_2 and then makes an abrupt jump back to V_{sub}. By assumption there is a positron occupying the bound state for if we had started with an initially vacant electron bound state (for a weak potential) then as the potential became stronger the bound state would have crossed $E = 0$ and then the absence of an electron in that state is now interpreted as the presence of a positron. The implicit assumption is that the initial electron vacancy (or equivalently positron prescence) persists during the switching on process from V_{weak} to V_{sub}. Wavefunctions pertaining to the supercritical potential and corresponding creation and annihilation operators will be denoted by a ($\sim$), the time argument of the latter ranging from t_1 to t_2.

The crucial observation is that although the potential may change abruptly at $t = t_1$ the field $\psi(x,t)$ is continuous. Thus we can expand $\psi(x,t_1)$ in terms of operators pertaining both to $t = t_1-$ and to $t = t_1+$. For $t = t_1-$ the expansions read

$$\psi(x,t_1) = \textstyle\sum_k \{a_e(k,t_1)u_{(+)e}(k,x) + a_o(k,t_1)u_{(+)o}(k,x) +$$

$$+c_e^\dagger(k,t_1)u_{(-)e}(k,x) + c_o^\dagger(k,t_1)u_{(-)o}(k,x)\} + d_{even}^\dagger(t_1)u_{even}(x) + b_{odd}(t_1)u_{odd}(t_1) \tag{6.1}$$

$$\psi^\dagger(x,t_1) = \textstyle\sum_k \{a_e^\dagger(k,t_1)u_{(+)e}^\dagger(k,x) + a_o^\dagger(k,t_1)u_{(+)o}^\dagger(k,x) +$$

$$+c_e(k,t_1)u_{(-)e}^\dagger(k,x) + c_o(k,t_1)u_{(-)o}^\dagger(k,x)\} + d_{even}(t_1)u_{even}(x) + b_{odd}^\dagger(t_1)u_{odd}(t_1) \tag{6.2}$$

and for $t = t_1+$

$$\psi(x,t_1) = \textstyle\sum_k \{\tilde{a}_e(k,t_1)\tilde{u}_{(+)e}(k,x) + \tilde{a}_o(k,t_1)\tilde{u}_{(+)o}(k,x) +$$

$$+\tilde{c}_e^\dagger(k,t_1)\tilde{u}_{(-)e}(k,x) + \tilde{c}_o^\dagger(k,t_1)\tilde{u}_{(-)o}(k,x)\} + \tilde{b}_{odd}(t_1)\tilde{u}_{odd}(t_1) \tag{6.3}$$

$$\psi^\dagger(x,t_1) = \textstyle\sum_k \{\tilde{a}_e^\dagger(k,t_1)\tilde{u}_{(+)e}^\dagger(k,x) + \tilde{a}_o^\dagger(k,t_1)\tilde{u}_{(+)o}^\dagger(k,x) +$$

$$+\tilde{c}_e(k,t_1)\tilde{u}_{(-)e}^\dagger(k,x) + \tilde{c}_o(k,t_1)\tilde{u}_{(-)o}^\dagger(k,x)\} + \tilde{b}_{odd}^\dagger(t_1)\tilde{u}_{odd}^\dagger(t_1) \tag{6.4}$$

Expansion (6.3) can be inverted to give

$$\tilde{a}_e(k,t_1) = \int dx \tilde{u}^\dagger_{(+)e}(k,x)\psi(x,t_1) \tag{6.5}$$

$$\tilde{c}^\dagger_e(k,t_1) = \int dx \tilde{u}^\dagger_{(-)e}(k,x)\psi(x,t_1) \tag{6.6}$$

Similar expressions hold for the odd operators and for the bound state ones. Expressions for the Hermitean conjugates can be obtained by taking the complex conjugates of the right hand sides of (6.5,6.6). We invoke continuity and substitute (6.3) in (6.5), (6.6). We notice that we get overlap integrals of the form $\int dx \tilde{u}^\dagger_{(\pm)e,o}(k,x)u_{(\pm)e,o}(k',x)$. Given that outside the well the wavefuctions are plane waves such integrals are proportional to $\delta_{kk'}\delta_{eo}$. We thus get

$$\tilde{a}_e(k,t_1) = A_k a_e(k,t_1) + B_k c^\dagger_e(k,t_1) + F_k d^\dagger_{even}(t_1) \tag{6.7}$$

$$\tilde{c}^\dagger_e(k,t_1) = G_k a_e(k,t_1) + L_k c^\dagger_e(k,t_1) + M_k d^\dagger_{even}(t_1) \tag{6.8}$$

where

$$A_k = \int dx \tilde{u}^\dagger_{(+)e}(k,x)u_{(+)e}(k,x), B_k = \int dx \tilde{u}^\dagger_{(+)e}(k,x)u_{(-)e}(k,x)$$

$$F_k = \int dx \tilde{u}^\dagger_{(+)e}(k,x)u_{even}(x),\ G_k = \int dx \tilde{u}^\dagger_{(-)e}(k,x)u_{(+)e}(k,x) \tag{6.9}$$

$$L_k = \int dx \tilde{u}^\dagger_{(-)e}(k,x)u_{(-)e}(k,x),\ M_k = \int dx \tilde{u}^\dagger_{(-)e}(k,x)u_{even}(x)$$

Expressions for $\tilde{a}^\dagger_e(k,t_1)$ and $\tilde{c}_e(k,t_1)$ are obtained by taking the Hermitean conjugates of (6.7), (6.8) respectively. The corresponding equations for the odd modes of course involve the odd rather than the even bound state because of parity

$$\tilde{a}_o(k,t_1) = A'_k a_o(k,t_1) + B'_k c^\dagger_o(k,t_1) + F'_k b_{odd}(t_1) \tag{6.10}$$

$$\tilde{c}^\dagger_o(k,t_1) = G^{'}_k a_o(k,t_1) + L'_k c^\dagger_o(k,t_1) + M'_k b_{odd}(t_1) \tag{6.11}$$

where the coefficients $A'_k, B'_k, F'_k, G'_k, L'_k, M'_k$ are defined as in (6.9) replacing *even* by *odd*. For the anticommutation relations

$$\{\tilde{a}_e(k,t), \tilde{a}^\dagger_e(k',t)\} = \delta_{kk'}$$

$$\{\tilde{c}_e(k,t), \tilde{c}^\dagger_e(k',t)\} = \delta_{kk'}$$

to be valid we should have

$$A_k^* A_k + B_k^* B_k + F_k^* F_k = 1$$

$$G_k^* G_k + L_k^* L_k + M_k^* M_k = 1$$

(no summation over k). Both of them are satisfied due to the fact that the set of wavefunctions $u_{(+)e}(k,x), u_{(-)e}(k,x), u_{even}(x)$ form an orthonormal set. Similar relations are valid for the primed coefficients. We now make some rather drastic approximations. Since the potential is assumed initially to be just subcritical the even state lies very near the negative energy continuum. Also the change in the potential is assumed to be small so that we end up with a slightly supercritical potential. Thus it is natural to assume that there is negligible overlap between the new positive energy wavefunctions on the one hand and the old negative energy wavefunctions and the bound state on the other. Hence in (6.9) $A_k = B_k = F_k = 0$ and

$$\tilde{a}_e(k,t_1) = a_e(k,t_1) \tag{6.12}$$

$$\tilde{a}_e^\dagger(k,t_1) = a_e^\dagger(k,t_1) \tag{6.13}$$

By the same token we can take $G_k = 0$. Thus

$$\tilde{c}_e(k,t_1) = M_k^* d_b(t_1) + L_k^* c_e(k,t_1) \tag{6.14}$$

$$\tilde{c}_e^\dagger(k,t_1) = M_k d_b^\dagger(t_1) + L_k c_e^\dagger(k,t_1) \tag{6.15}$$

Note that in the approximation we are working (neglecting the overlap integrals of the bound state with new states lying in the positive energy continuum), i.e. essentially assuming completeness of the set $\tilde{u}_{(-)e}(k,x)$, and using the orthogonality property of the above set we get

$$\sum_k M_k M_k^* = 1 \tag{6.16}$$

Following a similar reasoning we deduce that all odd operators referring to $t = t_1-$ are equal to the corresponding odd operators referring to $t = t_1+$.

Having obtained the relations beteen operators we come to the definition of the state $|S\rangle$ of the system in terms of operators pertaining to $t = t_1 - $. Recall that since we work in the Heisenberg picture the state vector is time independent however the vacuum (as all basis kets) is time dependent. The statement that there is one positron occupying the even state is equivalent to

$$d^\dagger(t_1) d(t_1) \, |S\rangle = |S\rangle \tag{6.17}$$

We also assume that there are no free electrons or positrons, i.e.

$$a_{e,o}(k,t_1) \, |S\rangle = c_{e,o}(k,t_1) \, |S\rangle = 0 \tag{6.18}$$

As already mentioned we can arrange the parameters so that when the first even state merges with the negative continuum only one (odd) bound state exists. Although it does not matter in what follows and simply in order to fix the notation we have to decide whether this odd state is above or below zero when the potential assumes its critical value. We take the first possibility; this is indeed the case for the value $a = .7$ mentioned in the previous section. We assume that there is an electron vacancy in the odd state, i.e.

$$b_{odd}(t_1)\,|S\rangle = 0 \tag{6.19}$$

Relations (6.17), (3.12), (3.13) specify the state completely. Using the above equations together with (6.14), (6.15) we can calculate the average number of positrons with momentum k for the supercritical potential:

$$N_{e,k} = \left\langle S|\tilde{c}_e^{\dagger}(k)\tilde{c}_e(k)|S\right\rangle = M_k M_k^* \tag{6.20}$$

(6.16) gives

$$\sum_k N_{e,k} = 1 \tag{6.21}$$

i.e. there is one positron emitted. Expression (6.20) for $N_{e,k}$ yields information on the energy spectrum of the positron emitted. Notice that the mixing between creation and annihilation operators at $t = t_1$, the ordering of the operators in (6.20) and the use of (6.16) are crucial.

It should be clear from the above discussion that if the original bound state is filled by an electron then

$$d(t_1)\,|S\rangle = 0$$

and the above reasoning leads to the conclusion that no positron is emitted.

We should check whether positron emission as described above is consistent with charge conservation. The Hamiltonian appropriate for the time dependent well considered here is given by (3.10) hence the total charge defined by (3.5) is indeed conserved. To demonstrate this we have to evaluate the charge expectation value $\langle S|\,Q\,|S\rangle$ with Q expressed in the form (3.6) in terms of the ($\sim$) operators pertaining to t_1+ and compare it to $\langle S|\,Q\,|S\rangle$ evaluated at $t = t_1-$. At t_1+

$$\begin{aligned} Q = \textstyle\sum_k\{&-\tilde{a}_e^{\dagger}(k,t_1)\tilde{a}_e(k,t_1) + \tilde{c}_e^{\dagger}(k,t_1)\tilde{c}_e(k,t_1) - \tilde{a}_o^{\dagger}(k,t_1)\tilde{a}_o(k,t_1) + \\ &+\tilde{c}_o^{\dagger}(k,t_1)\tilde{c}_o(k,t_1) - \tilde{b}_{odd}^{\dagger}(t_1)\tilde{b}_{odd}(t_1)\} + Q_{0,\sup er} \end{aligned} \tag{6.22}$$

and at t_1-

$$\begin{aligned} Q = \textstyle\sum_k\{&-a_e^{\dagger}(k,t_1)a_e(k,t_1) + c_e^{\dagger}(k,t_1)c_e(k,t_1) - a_o^{\dagger}(k,t_1)a_o(k,t_1) + \\ &+c_o^{\dagger}(k,t_1)c_o(k,t_1) - b_{odd}^{\dagger}(t_1)b_{odd}(t_1) + d_{even}^{\dagger}(t_1)d_{even}(t_1)\} + Q_{0,sub} \end{aligned} \tag{6.23}$$

Using (6.21) we notice that the increase of the $\tilde{c}^\dagger\tilde{c}$ contribution in the evaluation of the charge expectation value by one in (6.22) exactly balances the contribution of the $d^\dagger d$ in (6.23), thus the contribution of the normal ordered charge Q_{norm} rests unchanged. We also observe that $Q_{0,sub} = Q_{0,\sup er}$ when the potential becomes supercritical: An energy level crosses $-m$ but this does *not* change the spectral asymmetry. (The latter would change only in the case where the energy level crosses *zero* but this does not happen at $t = t_1$.) One can see this either from (3.8) by inspection or in more detail from (4.26): when a bound state merges with the negative energy continuum N_- decreases by 1 but $\delta(-m)$ increases by π, the net result being zero.

Regardless of the experimental feasibility we can in principle arrange for the potential to achieve the supercritical value *and then remain fixed*. We would then observe one positron emitted while V goes from V_{sub} to $V_{\sup er}$ and *nothing else*. Thus the transition from a subcritical to a supercritical potential creates just one positron in this case and not an electron-positron pair as Zeldovich and Popov (and others) predict. We shall see presently how and in what sense an additional electron appears.

At time t_1+ the positron charge density is concentrated inside the well and subsequently leaks away. Since the ex bound state has been transformed to a transmission resonance the characteristic time scale for the process is related to the corresponding time delay. Since we intend to switch back to a subcritical potential at time t_2 the time interval $t_2 - t_1$ should be rather large compared to the time delay so that the positron has time to escape. An estimate of the lifetime is provided in Appendix A.

When we switch back to the subcritical value the positron has escaped from the well and its charge is carried by travelling waves, hence the expectation value $\left\langle S|c_e^\dagger(k)c_e(k)|S\right\rangle$ has increased by 1 compared to its value for $t < t_1$. We have already shown that Q_0 stays unchanged when a level crosses the threshold $-m$. Thus charge conservation requires that there is *no* positron occupying the *even* bound state for $t > t_2$. When the potential returns to its original value V_{sub} the absence of a bound positron signifies the presence of a bound electron. Thus the round trip $V_{sub} \longrightarrow V_{\sup er}$ $\longrightarrow V_{sub}$ starting with an empty electron bound state results to the emission of one positron and the creation of one bound electron. It is only in that sense that one may talk about pair creation. It should be clear that the roles of positron and electron are rather unequal, the former being free and the latter being bound. Their roles would of course be reversed for a potential of a different sign. It must be stressed again that if the bound state were filled by a positron when the potential is slightly subcritical then *nothing* would happen when the potential becomes supercritical.

If one prepares the next (higher lying) odd bound state to be empty then as the potential becomes even stronger, this state merges with the negative energy continuum when

$$V = V_2^c = \sqrt{\frac{\pi^2}{a^2} + m^2} + m$$

and there is one further positron emitted. On the return trip the odd bound state is occupied by an electron.

7. Conclusions

We hope that we have now shown how to write down a second quantised theory of spontaneous fermion production in the context of a Dirac particle bound by an external field. Dirac's own understanding of a physical positron state as an unfilled electron state of negative energy is robust and perfectly capable of dealing with this phenomenon. Considerations of the vacuum charge (and the related topic of Levinson's theorem) are required for a full description of the problem. Zeldovich and Popov's intuition which allowed them to give a detailed analysis of spontaneous positron production based solely on the first quantised Dirac equation is generally sound. The one place it fails is in its identification of the cause of positron production with electron-positron pair production. This identification can be made when the supercritical potential is switched on at early times and off at late times, but when the potential is switched on and remains on then spontaneous positron production is distinct from electron-positron pair production.

We have not shown why narrow positron peaks are observed experimentally in heavy ion collisions [17]; in principle the overlap integral (6.20) taken between realistic Coulomb wave functions should provide a clue. A detailed knowledge of the time-dependence of the external potential is, however, probably more important. In this paper, we only assume a trivial time dependence in order to demonstrate the fundamental principles of spontaneous positron radiation, but we hope to return to the more general problem of treating electron-positron pair production in a time-dependent external field in a forthcoming paper.

A. Appendix A.

According to nonrelativistic quantum mechanics the time delay Δt of a wavepacket built around a transmission resonance is given by

$$\Delta t = \frac{1}{v_0}\frac{d\delta}{dk}$$

where v_0 is the group velocity of the wavepacket, k the wavevector corresponding to the resonance energy and δ the phase shift (see e.g. [10]). In the present case v_0 is the relativistic group velocity

$$v_0 = \frac{k}{|E|}\ ,\ E^2 - k^2 = m^2.$$

The phase shift is given by expression (2.19)

$$\delta(E) = \arctan\left(\frac{1+\gamma^2}{2\gamma}\tan 2pa\right) - 2ka \tag{A.1}$$

where

$$\gamma = \frac{k}{p}\frac{E+V+m}{E+m}$$

and $p = \sqrt{(E+V)^2 - m^2}$. We are interested in the $N = 1$ transmission resonance, hence p satisfies

$$2pa = \pi \tag{A.2}$$

From (A.1) we get

$$\frac{d\delta}{dk} = a\left(\gamma + \frac{1}{\gamma}\right)\frac{dp}{dk} - 2a \tag{A.3}$$

where we have used (A.2). Using the expressions for p and k in terms of E we readily calculate the derivative

$$\frac{dp}{dk} = \frac{k}{p}\frac{E+V}{E}$$

Then after some algebra

$$\left(\gamma + \frac{1}{\gamma}\right)\frac{dp}{dk} = \frac{E+V}{E}\left(\frac{E-m}{E+V-m} + \frac{E+m}{E+V+m}\right) \tag{A.4}$$

The above expression is exact. We now make the approximation of a just supercritical potential. Hence we can replace E by $-m$ and V by V_1^c as given by (2.31). Substituting (A.4) in (A.3) we deduce

$$\frac{d\delta}{dk} = 2a\frac{m}{\sqrt{\frac{\pi^2}{4a^2} + m^2} - m}$$

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